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THE LÉVY LAPLACIAN AND THE LÉVY PROCESS

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In this paper we shall discuss the Lévy Laplacian as an operator acting on some class of the Lévy functionals. We introduce some domain of the Laplacian as a Fock space associated with the Laplacian and give associated semigroups and associated stochastic processes.

Introduction

An infinite dimensional Laplacian was introduced by P. Lévy in his famous book [12]. Since then this exotic Laplacian has been studied by many authors from various aspects see [1–6,9,10,13,15,16,17,20] and references cited therein. In this paper, we discuss a stochastic process associated with the Lévy Laplacian generalizing the methods developed in the former works [11,14,18,22–26].

This paper is organized as follows. In Section 1 we give a necessary and sufficient conditon for a Fock space associated with the Lévy Laplacian. By this condition we summarize basic elements of white noise theory based on a stochastic process given as a difference of two independent Lévy processes in Section 2. In Section 3, following the recent works Kuo–Obata–Saitô [11], Obata–Saitô [18], Saitô [23,24] and Saitô–Tsoi [25], we formulate the Lévy Laplacian acting on a Hilbert space consisting of multiple Wiener integrals by the stochastic process. In Section 4, we generalize this situation by means of a direct integral of Hilbert spaces. In Section 5, based on infinitely many Cauchy processes, we give an infinite dimensional stochastic process associated with the Lévy Laplacian.

1. Background

Let $X = \{X(t);\ t \in \mathbf{R}\}$ be a Lévy process, of which the characteristic function is given by

$$E[e^{izX(t)}] = \exp\{th_X(z)\}, \quad z \in \mathbf{R},$$
$$h_X(z) = imz - \frac{\sigma^2}{2}z^2 + \int_{|\lambda|>0}\left(e^{iz\lambda} - 1 - \frac{iz\lambda}{1+\lambda^2}\right)\frac{1+\lambda^2}{\lambda^2}d\beta(\lambda),$$

where $\sigma \geq 0$,$m \in \mathbf{R}$ and β is a positive finite measure on $\mathbf{R}$ with $\beta(\{0\}) = \sigma^2$ and $\int_{\mathbf{R}} |\lambda|^n d\beta(\lambda) < +\infty$ for all $n \in \mathbf{N}$.

Let $E = \mathcal{S}(\mathbf{R})$ be the Schwartz space of rapidly decreasing $\mathbf{R}$-valued functions on $\mathbf{R}$. There exists an orthonormal basis $\{e_\nu\}_{\nu \geq 0}$ of $L^2(\mathbf{R})$ contained in E such that $Ae_\nu = 2(\nu+1)e_\nu$, $\nu = 0,1,2,\ldots$, with $A = -\frac{d^2}{du^2} + u^2 + 1$. Then by the Bochner-Minlos Theorem, there exists a probability measure μ on E^* such that

$$\int_{E^*} e^{i\langle x,\xi\rangle}d\mu(x) = \exp\{\int_{\mathbf{R}} h_X(\xi(t))dt\}, \qquad \xi \in E.$$

Let $L^2(E^*,\mu)$ be the Hilbert space of $\mathbf{C}$-valued square-integrable functions on (E^*,μ). The $\mathcal{U}$-transform $\mathcal{U}[\varphi]$ of $\varphi \in L^2(E^*,\mu)$ is defined by

$$\mathcal{U}[\varphi](\xi) = \exp\{-\int_{\mathbf{R}} h_X(\xi(t))dt\}\int_{E^*} e^{i\langle x,\xi\rangle}\varphi(x)d\mu(x), \ \xi \in E,$$

and the Wick product $\langle\cdot,f\rangle^{\diamond n}$ of $\langle\cdot,f\rangle$ is given by

$$\mathcal{U}[\langle\cdot,f\rangle^{\diamond n}] = \mathcal{U}[\langle\cdot,f\rangle]^n \ , \ f \in E.$$

Fixing a finite interval T on $\mathbf{R}$, we take an orthonormal basis $\{\zeta_n\} \subset E$ for $L^2(T)$ which is equally dense and uniformly bounded. Let $Dom(\Delta_L)$ denote the set of all $\varphi \in L^2(E^*,\mu)$ such that the limit

$$\widetilde{\Delta}_L\mathcal{U}[\varphi](\xi) \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(\mathcal{U}\varphi)''(\xi)(\zeta_n,\zeta_n),$$

exists for each $\xi \in E$ and a functional $\widetilde{\Delta}_L\mathcal{U}$ is in $\mathcal{U}[(L^2)]$. The Lévy Laplacian Δ_L on $Dom(\Delta_L)$ is defined by

$$\Delta_L\varphi = \mathcal{U}^{-1}\widetilde{\Delta}_L\mathcal{U}\varphi, \quad \varphi \in Dom(\Delta_L).$$

Let $F_n(\xi) = \mathcal{U}[\langle \cdot, f\rangle^{\diamond n}](\xi)$ with supp$f \subset T$. Then we can calculate that

$$F_1(\xi) = \int_{\mathbf{R}} f(t)\left(\mu + i\sigma^2\xi(t) + \int_{|u|>0}\left(e^{i\xi(t)u} - \frac{1}{1+u^2}\right)\frac{1+u^2}{u}d\beta(u)\right)dt;$$

$$\widetilde{\Delta}_L F_1(\xi) = -\frac{1}{|T|}\int_{\mathbf{R}} f(t)\int_{\mathbf{R}} u(1+u^2)e^{i\xi(t)u}d\beta(u)dt;$$

and

$$\begin{aligned}\widetilde{\Delta}_L F_n(\xi) &= nF_1(\xi)^{n-1}\widetilde{\Delta}_L F_1(\xi)\\ &= -\frac{n}{|T|}F_1(\xi)^{n-1}\int_{\mathbf{R}} f(t)\int_{\mathbf{R}} u(1+u^2)e^{i\xi(t)u}d\beta(u)dt.\end{aligned}$$

With these calculations, we have the following necessary and sufficient condition for a Fock space associated with the Lévy Laplacian.

Theorem 1.1. *The functionals F_n for all $n \in \mathbf{N} \cup \{0\}$ are eigenfunctions of $\widetilde{\Delta}_L$ if and only if the following holds:*

- $\beta = \sigma^2\delta_0$

or

- $\sigma = 0, \beta = a\delta_\lambda + (a - m\lambda)\delta_{-\lambda}$ *for some* $\lambda > 0$ *and* $a \geq 0$ *with* $a \geq m\lambda$.

2. Preliminaries

For $p \in \mathbf{R}$ define a norm $|\cdot|_p$ by $|f|_p = |A^p f|_{L^2(\mathbf{R})}$ for $f \in E$ and let E_p be the completion of E with respect to the norm $|\cdot|_p$. Then E_p becomes a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space E_p' is identified with E_{-p} by extending the inner product $\langle \cdot, \cdot\rangle$ of $L^2(\mathbf{R})$ to a bilinear form on $E_{-p} \times E_p$. It is known that $E = \operatorname{proj\,lim}_{p\to\infty} E_p$, and $E^* = \operatorname{indlim}_{p\to\infty} E_{-p}$.

The canonical bilinear form on $E^* \times E$ is also denoted by $\langle \cdot, \cdot\rangle$. We denote the complexifications of $L^2(\mathbf{R})$, E and E_p by $L^2_{\mathbf{C}}(\mathbf{R})$, $E_{\mathbf{C}}$ and $E_{\mathbf{C},p}$, respectively.

Let $\{L^1_{\sigma,a,\lambda}(t)\}_{t\geq 0}$ and $\{L^2_{\sigma,a,\lambda}(t)\}_{t\geq 0}$ be independent Lévy processes of which the characteristic functions are given by

$$E[e^{izL^j_{\sigma,a,\lambda}(t)}] = e^{th(z)}, \quad t \geq 0, \quad j = 1, 2,$$

$$h(z) = imz - \frac{\sigma^2}{4}z^2 + a\left(1 + \frac{1}{\lambda^2}\right)(e^{i\lambda z} - 1),$$

where $m \in \mathbf{R}, \sigma \geq 0, a \geq 0$, and $\lambda > 0$.

Set $\Lambda_{\sigma,a,\lambda}(t) = L^1_{\sigma,a,\lambda}(t) - L^2_{\sigma,a,\lambda}(t)$ for all $t \geq 0$. Then we have

$$\begin{aligned} E[e^{iz\Lambda_{\sigma,a,\lambda}(t)}] &= e^{t(h(z)+h(-z))} \\ &= \exp\left\{-t\frac{\sigma^2}{2}z^2 + ta\left(1 + \frac{1}{\lambda^2}\right)(e^{i\lambda z} + e^{-i\lambda z} - 2)\right\},\ t \geq 0. \end{aligned}$$

This characteristic function of $\Lambda_{\sigma,a,\lambda}(t)$ is corresponding to the form in Theorem 1.1 in the case of $m = 0$.

Set

$$C(\xi) = \exp\left\{\int_{\mathbf{R}} (h(\xi_1(u)) + h(-\xi_2(u)))du\right\},\ \xi = (\xi_1, \xi_2) \in E \times E.$$

Then by the Bochner-Minlos Theorem, there exists a probability measure $\mu_{\sigma,a,\lambda}$ on $E^* \times E^*$ such that

$$\int_{E^* \times E^*} \exp\{i\langle x, \xi\rangle\}\, d\mu_{\sigma,a,\lambda}(x) = C(\xi),\ \xi = (\xi_1, \xi_2) \in E \times E,$$

where $\langle x, \xi\rangle = \langle x_1, \xi_1\rangle + \langle x_2, \xi_2\rangle$, $x = (x_1, x_2) \in E^* \times E^*$, $\xi = (\xi_1, \xi_2) \in E \times E$.

Let $(L^2)_{\sigma,a,\lambda} \equiv L^2(E^* \times E^*, \mu_{\sigma,a,\lambda})$ be the Hilbert space of $\mathbf{C}$-valued square-integrable functions on $E^* \times E^*$ with L^2-norm $\|\cdot\|_{\sigma,a,\lambda}$ with respect to $\mu_{\sigma,a,\lambda}$. We call an element of $(L^2)_{\sigma,a,\lambda}$ the *Lévy functional*. The Wiener–Itô decomposition theorem says that:

$$(L^2)_{\sigma,a,\lambda} = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. The $\mathcal{U}$-transform of $\varphi \in (L^2)_{\sigma,a,\lambda}$ is defined by

$$\mathcal{U}\varphi(\xi) = C(\xi)^{-1} \int_{E^* \times E^*} \varphi(x) \exp\{i\langle x, \xi\rangle\} d\mu_{\sigma,a,\lambda}(x),\ \xi \in E \times E.$$

Theorem 2.1. [19] *(see also* [7,9,16]*) Let F be a complex-valued function defined on $E \times E$. Then F is a $\mathcal{U}$-transform of some Lévy functional in $(L^2)_{\sigma,a,\lambda}$ if and only if there exists a complex-valued function G defined on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ such that*

1) for any ξ and η in $E_{\mathbf{C}} \times E_{\mathbf{C}}$, the function $G(z\xi+\eta)$ is an entire function of $z \in \mathbf{C}$,

2) there exist nonnegative constants K and γ such that

$$|G(\xi)| \leq K \exp\left[\gamma|\xi|_0^2\right], \quad \forall\xi \in E_{\mathbf{C}} \times E_{\mathbf{C}},$$

3)

$$F(\xi) = G\left(i\frac{\sigma^2}{2}\xi_1 + i\frac{\sigma^2}{2}\xi_2 + a\left(1+\frac{1}{\lambda^2}\right)\lambda(e^{i\lambda\xi_1} - e^{-i\lambda\xi_2})\right)$$

for all $\xi = (\xi_1, \xi_2) \in E \times E$.

3. The Lévy Laplacian acting on the Lévy functionals

• Definition of the Lévy Laplacian

Consider $F = \mathcal{U}\varphi$ with $\varphi \in (L^2)_{\sigma,a,\lambda}$. By Theorem 2.1, for any $\xi, \eta \in E \times E$ the functions $z \mapsto F(\xi + z\eta)$ admits the Taylor series expansions:

$$F(\xi + z\eta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} F^{(n)}(\xi)(\eta, \ldots, \eta);$$

where $F^{(n)}(\xi) : E \times \cdots \times E \to \mathbf{C}$ is a continuous n-linear functional.

Fixing a finite interval T of $\mathbf{R}$, we take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset E \times E$ for $L^2(T)$ which is equally dense and uniformly bounded (see e.g. [9,10]). Let $\mathcal{D}_L$ denote the set of all $\varphi \in (L^2)_{\sigma,a,\lambda}$ such that the limit

$$\widetilde{\Delta}_L(\mathcal{U}\varphi)(\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (\mathcal{U}\varphi)''(\xi)(\zeta_n, \zeta_n),$$

exists for any $\xi \in E \times E$ and $\widetilde{\Delta}_L(\mathcal{U}\varphi)$ is in $\mathcal{U}[(L^2)_{\sigma,a,\lambda}]$. The Lévy Laplacian Δ_L is defined by $\Delta_L\varphi = \mathcal{U}^{-1}\widetilde{\Delta}_L\mathcal{U}\varphi$, $\varphi \in \mathcal{D}_L$.

• Multiple Wiener integrals by the Lévy process

Given $\sigma \geq 0, \lambda > 0, a \geq 0, n \in \mathbf{N}$ and $f \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$, we consider $\varphi \in (L^2)_{\sigma,a,\lambda}$ of the form:

$$\varphi = \int_{T^n} f(u_1, \ldots, u_n) d\Lambda_{\sigma,a,\lambda}(u_1) \cdots d\Lambda_{\sigma,a,\lambda}(u_n). \tag{2.1}$$

The $\mathcal{U}$-transform $\mathcal{U}\varphi$ of φ is given by

$$\mathcal{U}\varphi(\xi) = \int_{T^n} f(u_1, \ldots, u_n) \prod_{j=1}^{n} \Xi_{\sigma,a,\lambda}(\xi)(u_j) du_1 \ldots du_n, \ \xi \in E \times E,$$

where

$$\Xi_{\sigma,a,\lambda}(\xi)(u_j) = i\frac{\sigma^2}{2}\xi_1(u_j) + i\frac{\sigma^2}{2}\xi_2(u_j) + a\left(1 + \frac{1}{\lambda^2}\right)\lambda(e^{i\lambda\xi_1(u_j)} - e^{-i\lambda\xi_2(u_j)}).$$

For any $\sigma \geq 0,\ \lambda > 0, a \geq 0$, and $n \in \mathbf{N}$ let $\mathbf{E}_{\sigma,a,\lambda,n}$ denote the space of φ which admits an expression as in (2.1), where f belongs to $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ and $\operatorname{supp} f \subset T^n$.

Set $\mathbf{E}_{\sigma,a,\lambda,0} = \mathbf{C}$ for any $\sigma \geq 0, \lambda > 0, a \geq 0$. Then $\mathbf{E}_{\sigma,a,\lambda,n}$ is a closed linear subspace of $(L^2)_{\sigma,a,\lambda}$. Using a similar method as in [25], we get the following

Theorem 3.1. *For each $\sigma \geq 0,\ n \in \mathbf{N},\ \lambda > 0$ and $a \geq 0$ the Lévy Laplacian Δ_L becomes a scalar operator on $\mathbf{E}_{\sigma,0,\lambda,n} \cup \mathbf{E}_{0,a,\lambda,n}$ such that $\Delta_L\varphi = 0$ for all $\varphi \in \mathbf{E}_{\sigma,0,\lambda,n}$ and $\Delta_L\varphi = -\frac{n\lambda^2}{|T|}\varphi$ for all $\varphi \in \mathbf{E}_{0,a,\lambda,n}$.*

For $N \in \mathbf{N}$ and $\lambda > 0$ let $\mathbf{D}_N^{0,a,\lambda}$ be the space of $\varphi \in (L^2)_{0,a,\lambda}$ which admits an expression $\varphi = \sum_{n=1}^{\infty} \varphi_n,\ \varphi_n \in \mathbf{E}_{0,a,\lambda,n}$, such that $|||\varphi|||^2_{N,0,a,\lambda} = \sum_{n=1}^{\infty} \alpha_N^\lambda(n) \|\varphi_n\|^2_{0,a,\lambda} < \infty$, where $\alpha_N^\lambda(n) = \sum_{\ell=0}^{N} \left(\frac{n\lambda^2}{|T|}\right)^{2\ell}$. By the Schwartz inequality we see that $\mathbf{D}_N^{0,a,\lambda}$ is a subspace of $(L^2)_{0,a,\lambda}$ and becomes a Hilbert space equipped with the new norm $|||\cdot|||_{N,0,a,\lambda}$.

Moreover, in view of the inclusion relations:

$$(L^2)_{0,a,\lambda} \supset \mathbf{D}_1^{0,a,\lambda} \supset \cdots \supset \mathbf{D}_N^{0,a,\lambda} \supset \mathbf{D}_{N+1}^{0,a,\lambda} \supset \cdots,$$

we define

$$\mathbf{D}_\infty^{0,a,\lambda} = \operatorname*{proj\,lim}_{N \to \infty} \mathbf{D}_N^{0,a,\lambda} = \bigcap_{N=1}^{\infty} \mathbf{D}_N^{0,a,\lambda}.$$

Then Δ_L becomes a continuous linear operator defined on $\mathbf{D}_{N+1}^{0,a,\lambda}$ into $\mathbf{D}_N^{0,a,\lambda}$ satisfying $|||\Delta_L\varphi|||_{N,0,a,\lambda} \leq |||\varphi|||_{N+1,0,a,\lambda},\ \varphi \in \mathbf{D}_\infty^{0,a,\lambda},\ N \in \mathbf{N}$. Therefore Δ_L is a continuous linear operator on $\mathbf{D}_\infty^{0,a,\lambda}$. Moreover the operator Δ_L is a self-adjoint operator densely defined in $\mathbf{D}_N^{0,a,\lambda}$ for each $N \in \mathbf{N}$ and $\lambda > 0$.

For each $t \geq 0, \lambda > 0$ and $a \geq 0$ we consider an operator G_t^λ on $\mathbf{D}_\infty^{0,a,\lambda}$ defined by

$$G_t^\lambda\varphi = \sum_{n=1}^{\infty} e^{-tn\lambda^2/|T|}\varphi_n,\ \varphi = \sum_{n=1}^{\infty} \varphi_n \in \mathbf{D}_\infty^{0,a,\lambda}.$$

We also define G_t^0 on $(L^2)_{\sigma,0,\lambda}$ as an identity operator by

$$G_t^0\varphi = \varphi, \ \ \varphi \in (L^2)_{\sigma,0,\lambda}.$$

Theorem 3.2. [11,23] *Let $\lambda > 0$ and $a \geq 0$. Then the family of operators $\{G_t^\lambda; t \geq 0\}$ on $\mathbf{D}_\infty^{0,a,\lambda}$ is an equi-continuous semigroup of class (C_0) of which the infinitesimal generator is Δ_L.*

4. Extensions of the Lévy Laplacian

Let $d\nu(\lambda)$ be a finite Borel measure on $\mathbf{R}$ satisfying

$$\int_{(0,\infty)} \frac{d\nu(\lambda)}{\lambda^4} < \infty.$$

Fix $N \in \mathbf{N}$ and $a \geq 0$. Let $\mathfrak{D}_N^\sigma$ be the space of (equivalent classes of) measurable vector functions $\boldsymbol{\varphi} = (\varphi^\lambda)$ with $\varphi^\lambda = \sum_{n=1}^\infty \varphi_n^\lambda \in \mathbf{D}_N^{0,a,\lambda}$ for all $\lambda > 0$, and $\varphi^0 \in (L^2)_{\sigma,0,\lambda}$, such that

$$|||\boldsymbol{\varphi}|||_N^2 = \|\varphi^0\|_{\sigma,0,\lambda}^2 + \sum_{n=1}^\infty \int_{(0,\infty)} \|\varphi_n^\lambda\|_{0,a,\lambda,N}^2 \, d\nu(\lambda) < \infty. \tag{3.1}$$

Then $\mathfrak{D}_N^\sigma$ becomes a Hilbert space with the norm given in (3.1).

In view of the natural inclusion $\mathfrak{D}_{N+1}^\sigma \subset \mathfrak{D}_N^\sigma$ for $N \in \mathbf{N}$, which is obvious from construction, we define $\mathfrak{D}_\infty^\sigma = \operatorname{proj\,lim}_{N\to\infty} \mathfrak{D}_N^\sigma = \bigcap_{N=1}^\infty \mathfrak{D}_N^\sigma$.

The Lévy Laplacian Δ_L is defined on the space $\mathfrak{D}_\infty^\sigma$ by

$$\Delta_L \boldsymbol{\varphi} = (\Delta_L \varphi^\lambda), \ \ \boldsymbol{\varphi} = (\varphi^\lambda) \in \mathfrak{D}_\infty^\sigma.$$

Then Δ_L is a continuous linear operator from $\mathfrak{D}_\infty^\sigma$ into itself. Similarly, for $t \geq 0$ we define

$$G_t \boldsymbol{\varphi} = (G_t^\lambda \varphi^\lambda), \ \ \boldsymbol{\varphi} = (\varphi^\lambda) \in \mathfrak{D}_\infty^\sigma.$$

Then by Theorem 3.2 we have the following:

Theorem 4.1. *The family $\{G_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) on $\mathfrak{D}_\infty^\sigma$ whose generator is given by Δ_L.*

Remark: Let $\widetilde{G}_t$ be an operator defined on $\mathcal{U}[\mathfrak{D}_\infty^\sigma]$ by $\widetilde{G}_t = \mathcal{U} G_t \mathcal{U}^{-1}$, $t \geq 0$. Then by the above theorem, $\{\widetilde{G}_t;\ t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the operator $\widetilde{\Delta}_L$.

5. Associated infinite dimensional stochastic processes

• Space $E^{[0,\infty)}$

For $p \in \mathbf{R}$ let $E_p^{[0,\infty)}$ be the linear space of all functions $\lambda \mapsto \xi_\lambda \in E_p \times E_p$, $\lambda \geq 0$, which are strongly measurable. An element of $E_p^{[0,\infty)}$ is denoted by $\boldsymbol{\xi} = (\xi_\lambda)_{\lambda \geq 0}$. Equipped with the metric given by

$$d_p(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_{[0,\infty)} \frac{|\xi_\lambda - \eta_\lambda|_p}{1 + |\xi_\lambda - \eta_\lambda|_p} \, d\nu(\lambda), \ \boldsymbol{\xi} = (\xi_\lambda), \ \boldsymbol{\eta} = (\eta_\lambda),$$

the space $E_p^{[0,\infty)}$ becomes a complete metric space.

In view of $d_p \leq d_q$ for $p \geq q$, we introduce the projective limit space

$$E^{[0,\infty)} = \operatorname*{proj\,lim}_{p \to \infty} E_p^{[0,\infty)}.$$

• Space $\mathbf{C}^{[0,\infty)}$

Similarly, let $\mathbf{C}^{[0,\infty)}$ denote the linear space of all measurable function $\lambda \mapsto z_\lambda \in \mathbf{C}$ equipped with the metric defined by

$$\rho(\mathbf{z}, \mathbf{u}) = \int_{[0,\infty)} \frac{|z_\lambda - u_\lambda|}{1 + |z_\lambda - u_\lambda|} \, d\nu(\lambda), \ \mathbf{z} = (z_\lambda), \ \mathbf{u} = (u_\lambda).$$

Then $\mathbf{C}^{[0,\infty)}$ is also a complete metric space.

• Extension of the $\mathcal{U}$-transform

The $\mathcal{U}$-transform can be extended to a continuous linear operator on $\mathfrak{D}_\infty^\sigma$ by

$$\mathcal{U}\varphi(\boldsymbol{\xi}) = (\mathcal{U}\varphi^\lambda(\xi_\lambda))_{\lambda \geq 0}, \ \boldsymbol{\xi} = (\xi_\lambda)_{\lambda \geq 0} \in E^{[0,\infty)},$$

for any $\varphi = (\varphi^\lambda)_{\lambda \geq 0} \in \mathfrak{D}_\infty^\sigma$.

The space $\mathcal{U}[\mathfrak{D}_\infty^\sigma]$ is endowed with the topology induced from $\mathfrak{D}_\infty^\sigma$ by the $\mathcal{U}$-transform. Then the $\mathcal{U}$-transform becomes a homeomorphism from $\mathfrak{D}_\infty^\sigma$ onto $\mathcal{U}[\mathfrak{D}_\infty^\sigma]$. The transform $\mathcal{U}\varphi$ of $\varphi \in \mathfrak{D}_\infty^\sigma$ is a continuous operator from $E^{[0,\infty)}$ into $\mathbf{C}^{[0,\infty)}$. We denote the operator by the same notation $\mathcal{U}\varphi$.

• Associated stochastic process

Let $\{X_t^j\}, j = 1,2$, be independent Cauchy processes with t running over $[0,\infty)$, of which the characteristic functions are given by

$$\mathbf{E}[e^{izX_t^j}] = e^{-t|z|}, \ z \in \mathbf{R}, \ j = 1,2.$$

Take a smooth function $\eta_T \in E$ with $\eta_T = 1/|T|$ on T.

Set

$$Y_t^\lambda = (X_{\lambda t}^1 \eta_T, -X_{\lambda t}^2 \eta_T) \ \ \lambda \geq 0.$$

Define an infinite dimensional stochastic process $\{\mathbf{Y}_t; t \geq 0\}$ starting at $\boldsymbol{\xi} = (\xi_\lambda)_{\lambda \geq 0} \in E^{[0,\infty)}$ by

$$\mathbf{Y}_t = (\xi_\lambda + Y_t^\lambda)_{\lambda \geq 0}, \ t \geq 0.$$

Then this is an $E^{[0,\infty)}$-valued stochastic process and we have the following

Theorem 5.1. *If F is the $\mathcal{U}$-transform of an element in $\mathfrak{D}_\infty^\sigma$, we have*

$$\widetilde{G}_t F(\boldsymbol{\xi}) = E[F(\mathbf{Y}_t)|\mathbf{Y}_0 = \boldsymbol{\xi}], \ t \geq 0. \tag{4.1}$$

Proof. We first consider the case when $F \in \mathcal{U}[\mathfrak{D}_\infty^\sigma]$ is given by

$$F(\boldsymbol{\xi}) = (F^\lambda(\xi_\lambda))_{\lambda \geq 0},$$

$$F^0 \in \mathcal{U}[(L^2)_{\sigma,0,\lambda}],$$

$$F^\lambda(\xi_\lambda) = \left(a\lambda\left(1 + \frac{1}{\lambda^2}\right)\right)^n \int_{T^n} f(\mathbf{u}) \cdot \prod_{j=1}^n \left\{e^{i\lambda\xi_{1,\lambda}(u_j)} - e^{-i\lambda\xi_{2,\lambda}(u_j)}\right\} d\mathbf{u},$$

with $f \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$. Then we have

$$\begin{aligned}
&E[F(\mathbf{Y}_t)|\mathbf{Y}_0 = \boldsymbol{\xi}] = (E[F^\lambda(\xi_\lambda + Y_t^\lambda)])_{\lambda \geq 0} \\
&= \Bigg(F^0(\xi_0)\delta_{\lambda,0} + \left(a\lambda\left(1 + \frac{1}{\lambda^2}\right)\right)^n \int_{T^n} f(\mathbf{u}) E\Bigg[\prod_{j=1}^n \Big\{e^{i\lambda\xi_{1,\lambda}(u_j)} e^{i\frac{\lambda}{|T|}X_{\lambda t}^1 1_{(0,\infty)}(\lambda)} \\
&\quad - e^{-i\lambda\xi_{2,\lambda}(u_j)} e^{i\frac{\lambda}{|T|}X_{\lambda t}^2 1_{(0,\infty)}(\lambda)}\Big\}\Bigg] d\mathbf{u}\Bigg)_{\lambda \geq 0} \\
&= \left(e^{-tn\lambda/|T|} F^\lambda(\xi_\lambda)\right)_{\lambda \geq 0} = (\widetilde{G_t^\lambda} F^\lambda(\xi_\lambda))_{\lambda \geq 0} = \widetilde{G_t} F(\boldsymbol{\xi}).
\end{aligned}$$

Next let $F = \left(F^0\delta_{\lambda,0} + \sum_{n=1}^{\infty} F_n^{\lambda}\right)_{\lambda\geq 0} \in \mathcal{U}[\mathfrak{D}_{\infty}^{\sigma}]$. Then for ν-almost all $\lambda > 0$ and for any $n \in \mathbf{N}$, F_n^{λ} is expressed in the following form:

$$F_n^{\lambda}(\xi_\lambda) = \lim_{N\to\infty} \left(a\lambda\left(1+\frac{1}{\lambda^2}\right)\right)^n \int_{T^n} f_\lambda^{[N]}(\mathbf{u}) \prod_{j=1}^{n} \left\{e^{i\lambda\xi_{1,\lambda}(u_j)} - e^{-i\lambda\xi_{2,\lambda}(u_j)}\right\} d\mathbf{u}.$$

Since $F^0 \in \mathcal{U}[(L^2)_{\sigma,0,\lambda}]$ and $F_n^{\lambda} \in \mathcal{U}[\mathbf{D}_{\infty}^{0,a,\lambda}]$, there exist $\varphi^0 \in (L^2)_{\sigma,0,\lambda}$ and $\varphi_n^{\lambda} \in \mathbf{D}_{\infty}^{0,a,\lambda}$ such that $F^0 = \mathcal{U}[\varphi^0]$ and $F_n^{\lambda} = \mathcal{U}[\varphi_n^{\lambda}]$ for ν-almost all λ and each n. By the Schwarz inequality, we see that

$$\sum_{n=1}^{\infty} E[|F_n^{\lambda}(\xi_\lambda + Y_t^{\lambda})|] \leq \sum_{n=0}^{\infty} ||\varphi_n^{\lambda}||_{0,a,\lambda} E[||\varphi_{\xi_\lambda + Y_t^{\lambda}}||_{0,a,\lambda}]$$
$$\leq \left\{\sum_{n=1}^{\infty} \alpha_N^{\lambda}(n)^{-1}\right\}^{1/2} \left\{\sum_{n=1}^{\infty} \alpha_N^{\lambda}(n)||\varphi_n^{\lambda}||_{0,a,\lambda}^2\right\}^{1/2} E[||\varphi_{\xi_\lambda+Y_t^{\lambda}}||_{0,a,\lambda}] < \infty,$$

where $\varphi_{\xi_\lambda} = C(\xi_\lambda)^{-1} e^{i\langle\cdot,\xi_\lambda\rangle}$ for ν-almost all $\lambda \geq 0$ and each $N \in \mathbf{N}$. Therefore by the continuity of $\widetilde{G_t^{\lambda}}, \lambda \geq 0$, we get that

$$E[F(\boldsymbol{\xi} + \mathbf{Y}_t)] = \left(\sum_{n=1}^{\infty} E[F_n^{\lambda}(\xi_\lambda + Y_t^{\lambda})]\right)_{\lambda\geq 0} = \left(\sum_{n=1}^{\infty} \widetilde{G_t^{\lambda}} F_n^{\lambda}(\xi_\lambda)\right)_{\lambda\geq 0}$$
$$= \left(\widetilde{G_t^{\lambda}} \sum_{n=1}^{\infty} F_n^{\lambda}(\xi_\lambda)\right)_{\lambda\geq 0} = \widetilde{G_t} F(\xi).$$

Thus we obtain the assertion. □

For any $\eta \in E^{[0,\infty)}$ we define a translation operator T_η on $\mathfrak{D}_{\infty}^{\sigma}$ by

$$(\mathcal{U}T_\eta\varphi)(\boldsymbol{\xi}) = (\mathcal{U}\varphi)(\boldsymbol{\xi} + \eta).$$

Theorem 5.2. *For all φ in $\mathfrak{D}_{\infty}^{\sigma}$ we have*

$$G_t\varphi = E[T_{(Y_t^{\lambda})_{\lambda>0}}\varphi], \quad t \geq 0. \tag{4.2}$$

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